The secretary problem

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Given n candidates, what is the best strategy for choosing the best?

At first one may think that the probability goes to 0 as we increase n instead we will find that it is bounded and concretely $1/e$

Chosing a random candidate obviously yelds $1/n$ chances of choosing the best for a sample of n candidates. A better approach should be considered:

A good approach and as we will see the best, is a comparing strategy:
out of the first n candidates we rule out r of them. These will be discarded and used to compare with the next candidates. Then the first one that is better that all the r first ones will be chosen. Let's examine this strategy with some examples: for $n = 2$ we have that we discard the first one and we choose the

second one. thus we have $1/2$ of probability of choosing the best. the dummy base case

for $n = 3$ We are gonna use $r = 1$ as the comparing sample. then we examine probabilities

1. If the best one is the first one then we have 0 chances
2. If it is the second then 100%
3. If it is the third we will choose him only if the second is worse than the first. which is in 50% of the cases

All in all the probability is $P_3^1 = 1/3 * 1/2 + 1/3 = 3/6 = 0.5$ for $r = 1$ and $n = 3$ Let's see also $n = 4$ we set up $r = 2$ we examine the cases as before:

1. If the best ones lie in the first 2 then the probability is 0
2. If it is the third then it is 100
3. Then if it is the fourth we choose him if third was not better than the 2nd AND 1st. Which happens in 2/3 of the cases

Then the probability is $P_4^2 = 1/4 + 1/4 \cdot 1/2 + 1/4 \cdot 1/3 = 3/6 = 0.458$

Let's derive the general formula for P_n^r

$$P_n^r = \frac{1}{n} \sum_{i=r+1}^{n} \frac{r}{i-1} \tag{1}$$

We will first derive the optimal r for each n. By observing carefully (1) we can transform it into something to remind us a Riemann integral:

$$P_n^r = \frac{r}{n} \sum_{i=r+1}^{n} \frac{n}{i-1} \cdot \frac{1}{n}$$

We can set $t = \frac{i}{n}$ and $x = \frac{r}{n}$. Which then by turning $\lim_{n \to \infty}$ we obtain the following Riemann integral:

$$P_n^r \equiv P(x) = x \int_x^1 1/t \, dt$$

We can take the derivative of $P(x)$ to obtain the value that maximizes the probability.

$$P^{'}(x) = ln(x) + 1 = 0$$
$$x = \frac{1}{e}$$

It is curious to obtain the Euler constant $1/e = 0.36787...$

Then for each n the optimal r is the one that maximizes P_n^r

We can use the value found $x = e^{-1}$ to calculate the probability as n it tends to infinity:

$$P_n = \frac{1}{e} \int_{e^{-1}}^1 1/t \, dt = \frac{-1}{e} ln(e^{-1}) = \frac{1}{e}$$

So we have that the probability as we tend to infinity is $1/e = 0.36787....$ The probability of the secretary problem never goes under that.

r	n	r/n	P_n^r
1	1	1	1
1	2	0.5	0.5
1	3	0.333..	0.5
1	4	0.25	0.458
2	5	0.4	0.433
2	6	0.333..	0.428
2	7	0.285..	0.414
3	8	0.375	0.410
3	9	0.333..	0.406
3	10	0.3	0.3986
4	11	0.363..	0.3984
4	12	0.333..	0.3955
5	13	0.384..	0.3907